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OPTIMAL SYNTHESIS IN GRID SCHEMES FOR QUASI-CONVEX APPROXIMATION FUNCTIONS†

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A single grid algorithm which constructs the value function and the optimal synthesis, based on a local quasi-differential approximations of the Hamilton–Jacobi equation, is considered. The optimal synthesis is generated by the method of extremal translation in the direction of generalized gradients. The quasi-convex approximation functions, for which it is possible to use a linear dependence of the space-time steps for correct interpolation of the nodal optimal control values, thus substantially reducing the amount of computation, simplifying the finite-difference formulae and permitting the use of simple operators involving constructions of the method of least squares, are investigated.

We shall use the stability properties of the value function [1] and the principle of extremal translation [2] to construct the value function and the optimal synthesis in grid approximation schemes intended for the numerical solution of the Hamilton–Jacobi (HJ) equations. The generalized solution of the HJ equation, the value function of the problem of optimal guaranteed control, is usually non-differentiable, and the optimal synthesis is discontinuous on switching surfaces. The development of the theory of HJ equations for minimax solutions [3, 4] and for viscosity solutions [5, 6] enables non-differentiable functions to be used.

An algorithm for constructing the value function and optimal synthesis in a single grid scheme is described in [7, 8]. The values of the optimal control at grid nodes are constructed by the method of extremal translation [2, 9] in the direction of generalized gradients. The principle of extremal aiming in the direction of quasi-gradients defined by a Yosida–Moreau transformation has previously been discussed in [10].

In the general case, there are two constraints on grid approximation schemes for constructing the optimal synthesis. First, the correct finite-difference operator must be chosen. For instance, the correct operator for the guaranteed minimization problem is a minimax operator defined on local concave hulls, while the use of Lax–Friedrichs operators, operators with local convex or linear hulls, is problematic. Conversely, in the dual guaranteed maximization problem, a maximin operator defined on local convex hulls must be used. Second, unlike the problem of the convergence of approximation schemes with respect to the norm of the space of continuous functions, for which only a linear dependence of the space-time approximation steps is required, the approximation must be of higher order of smallness with respect to the space variables than the time variable. This is due to the fact that, for approximation schemes on space-time grids, the values of the value function, its generalized gradients and optimal controls are computed only at the grid nodes. However, the trajectories constructed might slip between the nodes, so that the values at intermediate points must be found by interpolation of the nodal values of the controls. Correct interpolation requires a "good" approximation not only of the value function, but also of the surfaces of discontinuity of the gradients (in the generalized sense with respect to the norm of the space of continuously differentiable functions), and this is achieved in the general case by having a higher order of approximation in space than in time.

Below we introduce the property of local quasi-convexity for the approximated functions, which relaxes the above constraints: the finite-difference formulae are greatly simplified and it is possible to use a linear dependence of the space-time steps. Simple approximation formulae of the method of least squares can be used to construct local linear hulls. The optimal control values are computed by the method of extremal translation in the direction of the gradients of linear functions. The values of the optimal controls at intermediate points are found by linear interpolation of the nodal values. We will show that this ensures that the approximation value function does not increase along the trajectory that is generated, and is therefore optimal. We consider the extension of the construction to the problem of guaranteed control with a discontinuous integral exponent.

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1. THE PROBLEM OF THE OPTIMAL GUARANTEED CONTROL

We consider the problem of constructing the optimal guaranteed synthesis $p = U^*(t, x)$ in the dynamical system

$$\dot{x} = f(t, x, p, q) = A(t, x) + B(t, x)p + C(t, x)q \tag{1.1}$$

$$t \in T = [0, \theta], \; x \in R^n, \; p \in P \subset R^p, \; q \in Q \subset R^q$$

for the terminal payoff functional

$$J(x(\cdot)) = \sigma(x(\theta)) \tag{1.2}$$

Here x is an n-dimensional vector of the system, p is the control action and q is a perturbation. The sets P and Q are convex compact sets. The functions $A(t, x)$, $B(t, x)$, $C(t, x)$ are Lipschitz-continuous with constant L and have sublinear growth, ensuring continuity of the solutions. The payoff function $x \to \sigma(x)$ satisfies the Lipschitz condition.

The right-hand side of system (1.1) has a saddle point, which uniquely defines the Hamiltonian $(t, x, s) \to H(t, x, s)$

$$H(t, x, s) = \langle s, A(t, x)\rangle + \min_{p \in P}\langle s, B(t, x)p\rangle + \max_{q \in Q}\langle s, C(t, x)q\rangle \tag{1.3}$$

The problem is solved by constructing an optimal strategy $(t, x) \to U^*(t, x)$ which realizes the value of the value function

$$w(t_*, x_*) = \min_{U} \max_{x(\cdot) \in X(t_*, x_*, U)} \sigma(x(\theta)) = \max_{V} \min_{y(\cdot) \in Y(t_*, x_*, V)} \sigma(y(\theta)) \tag{1.4}$$

The trajectories $x(\cdot)$ and $y(\cdot)$ are defined as the limits of stepwise motion (broken Euler lines) [1], generated from the initial position (t_*, x_*) by strategies $p = U(t, x)$, and realizations $q = q(t)$ or strategies $q = V(t, x)$ and realizations $p = p(t)$ respectively.

The value function w is Lipschitz-continuous and therefore differentiable almost everywhere. At points of differentiability, it satisfies the Bellman–Isaacs equation, a first-order partial differential equation of HJ type

$$\frac{\partial w}{\partial t}(t, x) + H\left(t, x, \frac{\partial w}{\partial x}(t, x)\right) = 0, \quad (t, x) \in T \times R^n \tag{1.5}$$

The determination of the value function w also involves satisfying the boundary condition

$$w(\theta, x) = \sigma(x), \quad x \in R^n \tag{1.6}$$

The main characteristics of the value function are its properties of stability [1], which ensure that the trajectories of the dynamical system (1.1) in its level sets (Lebesgue sets) survive—its graph is weakly invariant. The stability properties and boundary condition form necessary and sufficient conditions which the value function must satisfy.

The stability properties can be expressed in compact form within the framework of non-smooth analysis. They are given in terms of the derivatives with respect to direction in [3, 4], which introduces the concept of a minimax (non-differentiable) solution of the HJ equation identical with the value function. An equivalent definition of the viscosity solution, formulated in terms of subdifferentials, is obtained in the framework of the theory of first-order partial differential equations in [5, 6], where theorems of the existence, uniqueness and well-posedness of the solutions are proved.

We will give a definition of the generalized solution in terms of conjugate derivatives [11].

Definition. The Lipschitz-continuous function w is called a generalized (minimax) solution of the Cauchy problem for the HJ equation (1.5) if it satisfies boundary condition (1.6) and the pair of differential inequalities

$$\inf_{s \in R^n} \sup_{h \in R^n} (\langle s, h\rangle - \partial_- w(t, x)|(1, h) - H(t, x, s)) \geq 0 \tag{1.7}$$

$$\sup_{s \in R^n} \inf_{h \in R^n} (\langle s, h\rangle - \partial_+ w(t, x)|(1, h) - H(t, x, s)) \leq 0 \tag{1.8}$$

Inequality (1.7) expresses the property of u-stability, and (1.8) expresses the property of υ-stability of the function w. At points of differentiability of the function w, inequalities (1.7) and (1.8) become the HJ equation (1.5).

The constructions of straight lines and dual variables which appear in formulae (1.7) and (1.8) are used in the finite-difference operators to approximate the generalized solution of the HJ equation.

We define the compact domain $G_r \in T \times R^n$, in which an approximation scheme for the HJ equation (1.5) is to be constructed, by an invariance condition: if $(t_0, x_0) \in G_r$, then $(t, x_0 + (t - t_0)b) \in G_r$ for all $\{t \in T, \| b \| \leq r\}$. Here

$$r > K, \quad K = \max_{(t,x,p,q)\in G\times P\times Q} \|f(t, x, p, q)\| \tag{1.9}$$

is the maximum velocity of the system in the set G, which is strongly invariant with respect to the differential inclusion

$$x \cdot (t) \in F(t, x(t)), \quad F(\tau, y) = \{f(\tau, y, p, q): p \in P, q \in Q\}$$

2. OPTIMAL SYNTHESIS IN GRID SCHEMES

We will construct an optimal control procedure $(t,x) \to U^*(t,x)$ which solves the problem of minimizing the functional (1.2) using a finite-difference construction CU which is a direct consequence of the property of υ-stability (1.8)

$$v(x) = CU(t, \Delta, u)(x) = \min_{y\in O(x,K\Delta)} \min_{s\in D^*G(y)} \{\Delta H(t,x, s) + G(y) - \langle s, y - x\rangle\} \tag{2.1}$$

From the known approximation $y \to u(y)$ of the value function w, assigned at time $t + \Delta$, $(t + \Delta, y) \in D_r$, the operator CU constructs the approximation $x \to \upsilon(x)$ at time t, $(t, x) \in G_r$.

In (2.1) the symbol $y \to G(y)$ denotes a local concave hull $y \to G(y)$ of the function $y \to u(y)$ in the closed neighbourhood $\bar{O}(x, r\Delta)$ of the point x of radius $r\Delta$.

The set $D^*G(y)$ is a superdifferential of the function G

$$D^*G(y) = \{s \in R^n : G(\bar{y}) - G(y) \leq \langle s, \bar{y} - y\rangle, \ \bar{y} \in \bar{O}(x, r\Delta)\}, \quad y \in \bar{O}(x, K\Delta)$$

assigned in the closed neighbourhood $\bar{O}(x, K\Delta)$ of the point x of the radius $K\Delta$, $r > K$.

The operator CU can be represented as the minimax construction

$$\min_{p\in P} \max_{q\in Q} G(y(t, x, \Delta, p, q)) \tag{2.2}$$

computed on concave hulls $y \to G(y)$ and defined on elements of the Euler broken line $y(t, x, \Delta, p, q)$

$$y(t, x, \Delta, p, q) = x + \Delta(A(t, x) + B(t, x)p + C(t,x)q) \tag{2.3}$$

We will consider an idealized approximation scheme with finite-difference operator CU, in which we set the grid Γ only with respect to time

$$\Gamma = \{t_0 < t_1 < \ldots < t_N = \theta\}, \quad \Delta = t_{i+1} - t_1, \quad i = 0, 1, \ldots, N-1$$

We will assume that the values of the function $u(t, x)$ approximating the value function $w(t, x)$ are computed at all points $(t, x) \in G_r$, $t \in \Gamma$, that is

$$u(\theta, x) = \sigma(x) \tag{2.4}$$

$$u(t_i, x) = CU(t_i, \Delta, u(t_{i+1}, \cdot))(x), \ (t_i, x), (t_{i+1}, x) \in G_r, \ i = 0, \ldots, N-1 \tag{2.5}$$

We determine the optimal strategy values $U^* = U^*(t, x)$ with respect to the principle of extremal aiming in the direction of the generalized gradients—supergradients s^* of the local concave hull G of the function u

$$U^* = U^*(t, x) = \arg \min_{p\in P} \langle s^*, B(t, x)p\rangle \tag{2.6}$$

$$s^* = s^*(t, x, y^*) = \arg \min_{s \in D^* G(y^*)} \{\Delta H(t, x, s) + G(y^*) - \langle s, y^* - x \rangle\} \tag{2.7}$$

$$y^* = y^*(t, x) = \arg \min_{y \in O(x, K\Delta)} \min_{s \in D^* G(y)} \{\Delta H(t, x, s) + G(y) - \langle s, y - x \rangle\} \tag{2.8}$$

Note that in the single approximation scheme the values of the approximation function (AF) $u(t,x)$ and the optimal strategy $U^*(t,x)$ are computed in parallel.

We fix the initial position (t_0, x_0). Consider the Euler broken line

$$x(\cdot) = \{x(t, t_0, x_0, U^*, q(\cdot)), t \in \Gamma \cap T\} \tag{2.9}$$

generated by the strategy U^* (2.6) and an arbitrary perturbation $t \to q(t)$

$$x(t_{i+1}) = x(t_i + \Delta) = x(t_i) + \Delta(A(t_i, x(t_i)) +$$

$$+ B(t_i, x(t_i))U^* + C(t_i, x(t_i))q(t_i)), \quad t_i, t_{i+1} \in \Gamma, \quad x(t_0) = x_0$$

The strategy U^* ensures that along the trajectory $x(\cdot)$ (2.9) the values of the approximation function $u(t,x)$ are non-increasing, from which the following theorem can be deduced [12].

Theorem 2.1. For arbitrary accuracy parameter $\varepsilon > 0$, a division step Δ of Γ can be found such that for any initial position $(t_0, x_0) \in D_r$ and arbitrary perturbations $t \to q(t)$, the trajectory $x(\cdot)$ (2.9) generated by the strategy U^* (2.6) satisfies the inequalities

$$\sigma(x(\theta)) < u(t_0, x_0) + \varepsilon, \quad |w(t_0, x_0) - u(t_0, x_0)| < \varepsilon \tag{2.10}$$

In practice, the approximation procedure (2.4), (2.5) need only be performed at the grid nodes of GR rather than at every point $(t,x) \in G_r$. If the grid GR is uniform and rectangular

$$GR = \{(t, x) \in G_r : t \in \Gamma, \quad x = (m_1 e_1 + \dots + m_n e_n)h\} \tag{2.11}$$

$$m_i = 0, \pm 1, \pm 2, \dots, \quad i = 1, \dots, n$$

$$e_i = (e_i^1, \dots, e_i^n), \quad e_i^i = 1, \quad e_i^j = 0, \quad i = 1, \dots, n, \quad i \neq j$$

The values of the AF $u(t, y_j)$ are computed only at grid nodes $(t, y_j) \in GR$. Its values at points of G_r are found by linear interpolation according to the given simplicial subdivision of Ω with vertices at the nodes of GR

$$u(t, y) = \sum u(t, y_j), \ (t, y) \in D_r, \ (t, y_j) \in GR, \quad y = \sum \alpha_j y_j, \ \alpha_j \geqslant 0, \ \sum \alpha_j = 1 \tag{2.12}$$

Here and everywhere below the summation is taken from $j = 0$ to $j = n$.

There are convergence theorems [13] for approximation schemes with the linear interpolation (2.12) in the space of continuous functions. An estimate of the convergence is the square root $C\Delta^{1/2}$ of the division step Δ.

It is rather more difficult to find the values of the optimal strategy $U^*(t, y_j)$ by interpolation, because linear interpolation is not usually suitable in the case of discontinuous strategies. Piecewise-constant (copying) interpolations are then used

$$U^*(t, y) = U^*(t, y_0), \quad \|y_0 - y\| = \min_{(t, y_j) \in GR} \|y_j - y\| \tag{2.13}$$

To ensure acceptable accuracy using copying interpolation, the discretization step in the phase space (h) must be smaller than the time step (Δ)

$$h = \beta(\Delta)\Delta, \quad \lim_{\Delta \downarrow 0} \beta\Delta = 0 \tag{2.14}$$

If condition (2.14) holds, the theorem on the optimality of the copying strategy (2.6), (2.13) applies.

Theorem 2.2. For arbitrary accuracy parameter $\varepsilon > 0$, a subdivision step Δ of the grid Γ can be found and grid GR (2.11) chosen with high discretization of the phase variables (2.14) for which, for any initial position $(t_0, x_0) \in D_r$ and arbitrary perturbations $t \to q(t)$, the trajectory $x(\cdot)$ (2.9) generated by the imitation strategy U^* (2.6), (2.13) satisfies inequalities (2.10).

Condition (2.14) requires the use of huge grids GR and an enormous number of computations.

We shall consider the possibility of using "normal" grids in which the discretization steps h and Δ are linearly related (γ is a fixed constant)

$$h = \gamma\Delta \tag{2.15}$$

We introduce the following quasi-convexity condition. Suppose the AF $u(t,y)$ is convex, apart from an infinitely small quantity $\mu\Delta^{1+b}$, $b > 0$, $\mu > 0$ in regions of size $\nu\Delta$, so that the following conditions hold

$$\sum\alpha_j u(t,\ x_j)+\mu\Delta^{1+b} \geqslant u(t,\ \sum\alpha_j x_j),\ \ \alpha_j \geqslant 0,\ \ j=0,\ 1,\ \ldots,\ n,\ \ \sum\alpha_j = 1 \tag{2.16}$$

$$\|x_k - x_l\| \leqslant \nu\Delta,\ \ x_k,\ \ x_l \in R^n,\ \ k,\ \ l=0,\ 1,\ \ldots,\ n;\ \ \nu = n^{1/2}\gamma + 2K$$

In that case linear interpolation can be used to find the values $U^*(t, y_j)$, in exactly the same way as the values of the AF $u(t, y)$ (2.12)

$$U^*(t,\ y) = \sum_{j=0}^{n} \alpha_j U_j^*,\ \ U_j^* = U^*(t,\ y_j),\ \ y = \sum\alpha_j y_j,\ \ \alpha_j \geqslant 0,\ \ \sum\alpha_j = 1 \tag{2.17}$$

We have the following optimality principle [7, 8] for trajectory (2.9) generated by the strategy $U^*(t, y)$ (2.17) with linear interpolation of the nodal values $U^*(t, y)$, computed by the principle of external aiming in the direction of supergradients (2.6).

Lemma. The approximation function $u(t, y)$ is non-increasing along motions $x(\cdot)$ (2.9)

$$u(t_i,\ x(t_i)) \geqslant u(t_{i+1},\ x(t_{i+1})) - \mu\Delta^{1+b} - L_w n^{1/2} L\gamma\Delta^2 \tag{2.18}$$

Proof. By virtue of the convexity relation (2.16), the Lipschitz-continuity of the function $u(t,y)$ and the definition of the strategy U^* (2.6), (2.17), there are inequalities which guarantee that relation (2.18) holds

$$\begin{aligned}
&u(t_{i+1},\ x(t_{i+1})) = u(t_i+\Delta,\ x(t_i)+\Delta f(t_i, x(t_i),\ U^*,\ q(t_i)) = \\
&= u(t_i+\Delta,\ \sum\alpha_j\bar{x}_j - \Delta(\sum\alpha_j f(t_i,\ x_j,\ U_j^*,\ q(t_i)) - f(t_i,\ x(t_i),\ U^*,\ q(t_i)) \leqslant \\
&\leqslant \sum_{j=0}^{n} \alpha_j u(t_i+\Delta,\ \bar{x}_j) + \mu\Delta^{1+b} + L_w n^{1/2} L\gamma\Delta^2 \leqslant \\
&\leqslant u(t_i,\ x(t_i)) + \mu\Delta^{1+b} + L_w n^{1/2} L\gamma\Delta^2
\end{aligned}$$

Here

$$x(t_i) = \sum\ \alpha_j x_j,\ \ (t_i,\ x_j) \in GR,\ \ \alpha_j \geqslant 0,\ \ j=0,\ 1,\ \ldots,\ n,\ \ \sum\alpha_j = 1$$

$$\bar{x}_j = x_j + \Delta(A(t_i,\ x_j) + B(t_i,\ x_j)U_j^* + C(t_i,\ x_j)q(t_i))$$

We also have the relation

$$\|\bar{x}_k - \bar{x}_l\| \leqslant n^{1/2}\gamma\Delta + \Delta 2K \leqslant \nu\Delta,\ \ k,\ \ l=0,\ 1,\ \ldots\ n$$

It is clear from relation (2.18) that the strategy $U^*(t, y)$ (2.6), (2.17) is optimal.

Theorem 2.3. For the divisions of Γ and "normal" linear grids GR (2.15), let the *AF* $u(t, y)$ be quasiconvex: (2.16). Then for all initial positions (t_0, x_0) and any perturbations $\tau \to q(\tau)$ the trajectory $x(\cdot)$ (2.9), generated by the strategy U^* (2.16) with linearly interpolated nodal values (2.17), satisfies the inequalities

$$\sigma(x(\theta)) \leqslant u(t_0,\ x_0) + \varphi(\Delta) \tag{2.19}$$

$$\varphi(\Delta) \leqslant (\theta - t_0)(\mu\Delta^b + L_w n^{1/2} L\gamma\Delta),\ \ \lim_{\Delta\downarrow 0}\varphi(\Delta) = 0$$

$$|w(t_0,\ x_0) - u(t_0,\ x_0)| \leqslant C\Delta^{1/2}$$

Having chosen the arbitrary number $\varepsilon > 0$, we can find a step Δ for Γ and a "normal" linear grid GR (2.15) for which (2.10) holds.

Remark. 2.1. The formulae in the finite-difference operator CU (2.1) for local convex hulls $y \to G(y)$ and superdifferentials $D^*G(y)$ are considerably simpler in a "normal" grid with linearly-dependent discretization steps of the phase variables and time (2.15)

$$CU(t,\ \Delta,\ u)(x) = G(x) + \Delta \min_{s \in D^* G(x)} H(t,\ x,\ s)$$

$$G(x) = \max\{u(x),\ \max\{(u(x+\gamma\Delta e_i) - (u(x-\gamma\Delta e_i))/2\}\}$$

$$D^*G(x) = \mathrm{co}\{b_k:\ k = 1,\ \dots,\ 2^n\},\ \ b_k = (b_k^1,\ \dots,\ b_k^n)$$

$$b_k^i = \pm\frac{G(x \pm \gamma\Delta e_i) - G(x)}{\gamma\Delta} = \pm\frac{u(x \pm \gamma\Delta e_i) - G(x)}{\gamma\Delta}$$

The principle of extremal aiming for the optimal strategy U^* is applied using the simple formulae

$$U^*(t,\ x) = \arg\min_{p \in P}\langle s^*,\ B(t,\ x)p\rangle,\ \ s^*(t,\ x) = \arg\min_{s \in D^* G(x)} H(t,\ x,\ s)$$

Remark 2.2. The quasi-convexity condition (2.16) gives an optimal strategy (2.6) on a "normal" grid with the simplest finite-difference operator LA, using local linear approximations from the method of least squares

$$LA(t,\ \Delta,\ u)(x) = u_0 + \Delta H(t,\ x,\ c)$$

$$u_0 = \frac{1}{M}\sum[u(y_0 + u(y_1) + \dots + u(y_M))],\ \ M = 2n,\ \ y_0 = x,\ \ y_i = x \pm \gamma\Delta e_i,\ \ i = 1,\ \dots,\ n$$

$$c = (c^1,\ \dots,\ c^n),\ \ c^i = \frac{u(x+\gamma\Delta e_i) - u(x-\gamma\Delta e_i)}{2\gamma\Delta},\ \ i = 1,\ \dots,\ n$$

$$U^*(t,\ x) = \arg\min_{p \in P}\langle c,\ B(t,\ x)p\rangle$$

3. A DIFFERENTIAL GAME WITH DISCOUNTING

We will consider a stationary control system in an infinite time interval $[0, +\infty)$

$$\dot{x} = f(x,\ p,\ q) = A(x) + B(x)p + C(x)q,\ \ x \in R^n,\ \ p \in P \subset R^p,\ \ q \in Q \subset R^q \tag{3.1}$$

Let $x(\cdot) = \{x(t): t \in [0, +\infty)\}$ be a trajectory of system (3.1) generated by samples $t \to p(t), t \to q(t)$ of the parameters p and q. As a measure of the quality of the process $(x(\cdot), p(\cdot), q(\cdot))$ we use an integral functional with discounting coefficient $\lambda > 0$

$$J(x(\cdot),\ p(\cdot),\ q(\cdot)) = \int_0^{+\infty} e^{-\lambda\tau} g(x(\tau),\ p(\tau),\ q(\tau))d\tau \tag{3.2}$$

The functions $f(\cdot), g(\cdot)$ in dynamical system (3.1) and integral functional (3.2) are continuous over the set of variables, satisfy a Lipschitz condition with respect to the variable x with constant L and are bounded by the constant K.

In problem (3.1), (3.2) the upper value function w^0 is defined by the relation

$$w^0(x_*) = \min_U \max_{(x(\cdot), z(\cdot)) \in Y(y_*, U)} \lim_{\theta \to \infty} z(\theta) \tag{3.3}$$

Here $Y(y_*, U)$ is the set of trajectories $y(t) = (x(t), z(t)), t \in [0, \theta]$ of the expanded system

$$\dot{x} = f(x,\ p,\ q),\ \ \dot{z} = e^{-\lambda t} g(x,\ p,\ q) \tag{3.4}$$

which are generated by the positional strategy $p = U(t, x)$ and arbitrary samples $q = q(t)$ from the initial position y_*

$$y_* = (x_*,\ z_*),\ \ x(0) = x_*,\ \ z(0) = z_* = 0$$

We know [14, 15] that the value function $x \to w^0(x)$ is Hölder constant ρ, depending only on the Lipschitz constant L and discounting coefficient λ, and that the boundedness condition is satisfied over the entire space R^n with constant K/λ. These conditions and stability properties give necessary and sufficient conditions which must be satisfied by the value function w^0. The stability properties can be expressed in infinitesimal form by the use of constructions of non-smooth analysis—derivative with respect to direction and conjugate derivatives. At points of differentiability of the value function w^0, the corresponding differential inequalities become the stationary HJ equation

$$-\lambda w^0 + H(x,\ \partial w^0/\partial x) = 0, \quad x \in R^n \tag{3.5}$$

The function $H(x, s)$: $R^n \times R^n \to R$ of Eq. (3.5) is the Hamiltonian of problem (3.1), (3.2) and is associated with the dynamics $f(x, p, q)$ and the integral function $g(x, p, q)$ by the relation

$$\begin{aligned} H(x,\ s) &= \min_{p\in P}\ \max_{q\in Q}\{\langle s,\ f(x,\ p,\ q)\rangle + g(x,\ p,\ q)\} = \\ &= \langle s,\ A(x)\rangle + \min_{p\in P}\ \max_{q\in Q}\{\langle s,\ B(x)p + C(x)q\rangle + g(x,\ p,\ q)\} \end{aligned} \tag{3.6}$$

We will consider an ideal iterative procedure with the values of the iteration functions $u^i_\Delta(x)(i = 0, \ldots, m)$, approximating the value function $w^0(x)$ assumed to be constructed at all points $x \in R^n$

$$\begin{aligned} &u^0_\Delta(x) = 0,\quad u^i_\Delta(x) = CUS(u^{i-1}_\Delta)(x),\quad i = 0,\ \ldots,\ m,\quad m = \theta/\Delta,\quad x \in R^n \\ &CUS(u^{i-1}_\Delta)(x) = \min_{y\in \bar{O}(x,K\Delta)} \varkappa_{i-1}(x,\ y) \\ &\varkappa_{i-1}(x,\ y) = \min_{s\in D^*(G_{i-1}(y))}\{\Delta H(x,\ e^{-\lambda\Delta}s) + e^{-\lambda\Delta}G_{i-1}(y) - \langle e^{-\lambda\Delta}s,\ y - x\rangle\} \end{aligned} \tag{3.7}$$

Here $G_{i-1}(y)$ is the local concave hull of the iteration function $u^i_\Delta(y)$, $y \in \bar{O}(x, r\Delta)$, $r > K$. The set $DG_{i-1}(y)$ is the superdifferential of the local concave hull $G_{i-1}(i = 0, \ldots, m)$ at the point $y \in \bar{O}(x, K\Delta)$.

We will define the value of the positional control $U^* = U^*(x)$ at each point $x \in R^n$ according to the principle of extremal aiming in the direction of the supergradient s^* of the local concave hull G_m of the iteration function u^m_Δ

$$\begin{aligned} &U^*(x) = \arg H(x,\ s^*) = \arg \min_{p\in P}\{\langle s^*,\ B(x)p\rangle + \max_{q\in Q}\{\langle s^*,\ C(x)q\rangle + g(x,\ p,\ q)\}\} \\ &s^* = s^*(x,\ y^*) = \arg \varkappa_m(x,\ y^*),\quad y^* = y^*(x) = \arg \min_{y\in\bar{O}(x,K\Delta)} \varkappa_m(x,\ y) \end{aligned} \tag{3.8}$$

The values of the iteration functions u^m_Δ are "non-increasing" along the trajectory $x(\cdot)$ generated by the strategy U^* (3.8) and the arbitrary perturbation $\tau \to q(\tau)$

$$\begin{aligned} &x(\cdot) = \{x(t,\ x_0,\ U^*,\ q(\cdot)),\quad t \in [0,\ +\infty)\} \\ &x(t_{i+1}) = x(t_i) + \Delta(A(x(t_i)) + B(x(t_i))U^* + C(x(t_i))q(t_i)) \\ &t_0 = 0,\quad t_{i+1} = t_i + \Delta,\quad x(0) = x_0 \end{aligned} \tag{3.9}$$

from which we obtain the following result.

Theorem 3.1. For any accuracy $\varepsilon > 0$, a step Δ and iteration m can be found for which, for any initial position $x_0 \in R^n$ and any perturbations $t \to q(t)$, the trajectory $x(\cdot)$ (3.9) generated by the strategy U^* (3.8) satisfies the inequalities

$$J(x(\cdot),\ U^*,\ q(\cdot)) < u^m_\Delta(x_0) + \varepsilon,\quad |w^0(x_0) - u^m_\Delta(x_0)| < \varepsilon \tag{3.10}$$

There is an analogous result for an approximation scheme with high-order discretization GS of the phase variables (2.14)

$$GS = \{x \in R^n : x = (m_1e_1 + \ldots + m_ne_n)h\}$$

and copying interpolation

$$U^*(y) = U^*(y_0), \quad \|y_0 - y\| = \min_{y_j \in GS} \|y_j - y\|$$

of the strategy U^* constructed at the nodes of GS by the method of extremal translation (3.8) on supergradients of the iteration function u^m_Δ.

Theorem 3.2. For any accuracy $\varepsilon > 0$, a step Δ, a grid GS with highly discretized phase variables (2.14), and an iteration m can be found for which, for any initial position $x_0 \in R^n$ and arbitrary perturbations $t \to q(t)$, the trajectory $x(\cdot)$ (3.9) generated by the strategy U^* (3.8) satisfies relations (3.10).

If the quasi-convexity condition (2.16) is satisfied by the iteration functions u^m_Δ in grids GS with a "normal" linear dependence (2.15) of steps h and Δ, the trajectories $x(\cdot)$ generated by the strategy U^* (3.8) with linearly interpolated nodal values $U^*(y_j)$ can have optimal properties

$$U^*(y) = \sum \alpha_j U_j^*, \quad U_j^* = U^*(y_j), \quad y = \sum \alpha_j y_j, \quad \alpha_j \geqslant 0, \quad \sum \alpha_j = 1 \tag{3.11}$$

Theorem 3.3. For "normal" linear grids GS, let the iteration functions $u^m_\Delta(y)$ possess the quasi-convexity properties (2.16). Then for all initial positions x_0 and any perturbations $\tau \to q(\tau)$, the trajectory $x(\cdot)$ (3.9) generated by the strategy U^* (3.8) with linearly-interpolated nodal values (3.11) satisfies the relations

$$J(x(\theta), U^*, q(\cdot)) \leqslant u^m_\Delta(x_0) + \theta(\mu\Delta^b + e^{(L-\lambda)\theta}\Delta) + K\lambda^{-1}e^{-\lambda\theta} \leqslant u^m_\Delta(x_0) + B\Delta^{\rho b}, \quad \theta = m\Delta$$

$$|w^0(x_0) - u^m_\Delta(x_0)| \leqslant C\Delta^{\rho/2}, \quad (1-\lambda\delta)^m \leqslant \Delta^{\rho/2}$$

Fixing an arbitrary number $\varepsilon > 0$, we can find a step Δ, a "normal linear grid" GS (2.15) and an iteration m for which relations (3.10) hold.

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